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Rees Algebras and Koszul Homology

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This paper deals with properties of a Cohen–Macaulay ideal I in a regular local ring R. We consider two main kinds of ideals: strongly Cohen–Macaulay ideals and ideals with sliding depth. One of our aims is to sharpen their differences. We also examine certain kinds of normal ideals and consider ways of determining the divisor class group and the canonical module of its Rees algebra. We show that the divisor class group of $R(I)$ is a finitely generated free abelian group. The main results of this work are certain numerical constraints that are shown to exist on the projective resolutions of graded ideals with Cohen–Macaulay Koszul homology.

INTRODUCTION

We study here structural aspects of the theory of Cohen–Macaulay ideals that are reflected by the corresponding Koszul homology. More specifically, we examine properties that are shared across the equivalence class of a Cohen–Macaulay ideal I under the relation induced by linkage. Among these the most directly available concern the depths of the Koszul homology modules of I. Two such notions, intimately related, are the *strongly Cohen–Macaulay* ideals and the ideals with *sliding depth*. They are both highly desirable properties because their presence facilitates the study of arithmetical and analytical properties as expressed, for instance, in the associated graded rings of I. Thus the examination of the normality of I or the Cohen–Macaulayness of the blowing-up ring of I is much simplified under such conditions. Furthermore, it reveals a large number of new Cohen–Macaulay and Gorenstein ideals with "exotic" properties.

We shall now describe in some detail the content of this paper. In the early portion of Section 1 we introduce our main objects of study and recall some notions of the theory of Cohen–Macaulay ideals. Because of the

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existence of excellent references for the basic theory of Noetherian rings and the relevant homological algebra this task is not hard. We then proceed to sharpen the comparison between ideals with sliding depth and strongly Cohen–Macaulay ideals as carried out in [16].

In Section 2 we examine the Rees algebras of the ideals of the kind studied in Section 1. More concretely, we are interested in those ideals that in addition are normal and consider ways of computing the divisor class group of its Rees algebra. The main result asserts that if the base ring R is a regular factorial ring then the divisor class group of $R(I)$ is a finitely generated free abelian group.

The last section has a different character. We introduce a bit of notation here. Let R be the polynomial ring in several variables over a field and let I be a homogeneous Cohen–Macaulay ideal. The simplest such ideals to study are those that are generated by forms of the same degree and whose higher syzygies are also generated by elements of the same degree. Thus, ideals of this kind and height 3 have resolution whose numerical data look like

$$0 \to R^{b_3}(-d-a-b) \to R^{b_2}(-d-a) \to R^{b_1}(-d) \to R \to R/I \to 0.$$

Such resolutions are known as *pure*. Here the Betti numbers b_i of the resolution are determined by the degree of the minimal generators of the syzygies (cf. [8]). There are several classes of examples of such ideals. We have attempted to explain the scarcity of strongly Cohen–Macaulay ideals outside of some very well circumscribed cases. Thus ideals of deviation at most 2 (that is, $b_1 \leqslant 5$) and Gorenstein ideals (that is, $b_3 = 1$) are always strongly Cohen–Macaulay regardless of any restriction on the resolution.

We conjecture that outside of all the known cases, Cohen–Macaulay ideals which are generically complete intersection and have pure resolution are never strongly Cohen–Macaulay. We have not, however, succeeded in establishing this fully. (Perhaps half of all possible cases have been eliminated.) Furthermore, another drawback of the "explanation" is that it is purely numerical, resulting from the examination of all possible Hilbert polynomials of the ring R/I. Also, the computations were all carried out by machine, although here it is quite possible—once the results are shown likely—to do it by hand.

The starting point of our analysis is a criterion of [19] which proved that, in the situation of the resolution above, if $d < a + b$ then I is a Gorenstein ideal. In turn, we show that if $a \geqslant b$ and $b_1 > 5$, then the one-dimensional Koszul homology module is not Cohen–Macaulay. This effectively takes care of all ideals with $b_1 \neq 8$ and $6 \leqslant b_1 \leqslant 24$. Furthermore in the only case known to us with $b_1 = 8$, we have verified that $H_1(I)$ is Cohen–Macaulay, but $H_2(I)$ is not Cohen–Macaulay.

1. Koszul Homology

Throughout all rings are commutative and Noetherian, and modules are finitely generated. For basic results, terminology, and notation—especially those dealing with Koszul complexes and Cohen–Macaulay rings—we shall refer to [20, 18]. To simplify the notation, tensor products and derived functors formulae shall be unadorned as long as the base ring is well understood.

Let I be an ideal of R, and let $H_1(I)$ be the first homology module of the Koszul complex $K_*(\mathbf{x}, R)$ associated to a set $\mathbf{x}$ of generators of I. In [21] it is pointed out that $H_1(I)$ is related to I/I^2, the conormal module of I, by the exact sequence

$$H_1(I) \xrightarrow{f} R^n \otimes (R/I) \xrightarrow{h} I \otimes (R/I) = I/I^2 \to 0.$$

Here $f(z) = z \otimes 1$ and $h(e_i \otimes 1) = x_i \otimes 1$.

(1.1) Definition. Set $\delta(I) = \operatorname{Ker}(f)$; we say that I is *syzygetic* if $\delta(I) = 0$.

(1.2) Definition. Let I be an ideal in R; we say that I is *generically a complete intersection* if I is unmixed and IR_p is a complete intersection for all $p \in \operatorname{Ass}_R(R/I)$. Let $n \geqslant 2$ be an integer; we define the symbolic powers $I^{(n)}$ of the ideal I by $I^{(n)} = I^n R_S \cap R$, where S is the complement of the union of the associated prime ideals of I.

Another notion that will play a role here is that of a d-sequence. First introduced by Huneke (cf. [14]), d-sequences are a generalization of "regular sequences."

(1.3) Definition. Let $\mathbf{x} = \{x_1, ..., x_n\}$ be a sequence of elements in a ring R. We say that $\mathbf{x}$ is a *d-sequence* if

(a) The ideal $I = (\mathbf{x})$ is minimally generated by $\mathbf{x}$.

(b) $((x_1, ..., x_i) : x_{i+1}x_k) = ((x_1, ..., x_i) : x_k)$, for $0 \leqslant i \leqslant n-1$ and $k \geqslant i+1$.

In the sequel (R, m) will be a CM (Cohen–Macaulay) local ring. Let $I = (x_1, ..., x_n)$ be an ideal of R. We denote by $H_*(\mathbf{x})$ the homology of the ordinary Koszul complex built on the sequence $\mathbf{x}$.

We now come to our major definition.

(1.4) DEFINITION. (i) (SD). We say that I satisfies *sliding depth* if depth $H_i(\mathbf{x}) \geqslant \dim(R) - n + i$, $i \geqslant 0$.

(ii) (SCM). We say that I is a *strongly Cohen–Macaulay* ideal if $H_i(\mathbf{x})$ are Cohen–Macaulay, for all $i \geqslant 0$.

(Depths are computed with respect to maximal ideals. As usual, we set depth $(0) = \infty$.)

It is convenient to rephrase the condition (SD) for an ideal I in terms of the depths of the cycles and boundaries of the associated Koszul complex. Assume R is a CM local ring of dimension d and I is generated by the sequence $\mathbf{x} = \{x_1, ..., x_n\}$; we set $g = \mathrm{ht}(I)$. Denote by Z_i and B_i the modules of cycles and boundaries of the associated Koszul complex $K_*(\mathbf{x})$. If one uses the defining exact sequences

$$0 \to Z_{i+1} \to K_{i+1} \to B_i \to 0$$

and

$$0 \to B_i \to Z_i \to H_i \to 0$$

the depth conditions (SD) and (SCM) translate as

$$\mathrm{depth}(Z_i) \geqslant \begin{cases} \min\{d, d-n+i+1\}, & \text{for (SD)} \\ \min\{d, d-g+2\}, & \text{for (SCM).} \end{cases}$$

The case $i = n - g$ was examined in [16], where the following result is proved.

(1.5) PROPOSITION. *Let R be a local Gorenstein ring of dimension d and let $I = (x_1, ..., x_n)$ be a CM ideal of height g. Then* $\mathrm{depth}(Z_{n-g}) \geqslant \min\{d, d-g+2\}$.

(1.6) DEFINITION. Let I be an ideal in R, of height g. The *deviation* of I is the deficit $\delta = n - g$, where $n = v(I) =$ minimum number of generators of I.

(1.7) *Remark.* (a) A bit more can be said in the last proposition: If the deviation of I is positive and its height is at least two, then $\mathrm{depth}(Z_{n-g}) = d - g + 2$. To check this we could use the tail of the Koszul complex and duality to get the isomorphism $\mathrm{Ext}^{g-2}(Z_{\delta-k}, R) \cong H_{k+1}(I)$, $k \geqslant 0$.

(b) Note that because $\mathrm{depth}(B_{\delta-1}) = d - g + 1$, in this case the sequence

$$0 \to B_{\delta-1} \to Z_{\delta-1} \to H_{\delta-1} \to 0$$

says that $\mathrm{depth}(H_{\delta-1}) \geqslant 2$. Therefore $H_{\delta-1}$ is an S_2-module.

Let R be a local Gorenstein ring, and let $I=(x_1, ..., x_n)$ be an ideal of R. We set $g=\operatorname{ht}(I)$ and $\delta=n-g$. Recall that $\operatorname{Ext}^g(S, R)\cong H_\delta(\mathbf{x})\cong W_s=$ the canonical module of $S=R/I$ (cf. [7]). In [5] the following duality was essentially proved: If $H_0(\mathbf{x}), ..., H_k(\mathbf{x})$ are CM, then so are $H_\delta(\mathbf{x}), ..., H_{\delta-k}(\mathbf{x})$.

Using essentially the same arguments used in [16] to prove Proposition (1.5), and also making use of the duality in the Koszul complex, the following result is not hard to prove.

(1.8) PROPOSITION. *Let R be a local Gorenstein ring of dimension d, and let $I=(x_1, ..., x_n)$ be an ideal of height $g\geqslant 2$. If $H_0(I), ..., H_r(I)$ are Cohen–Macaulay then* $\operatorname{depth}(Z_{\delta-k})=d-g+2$ *for* $0\leqslant k\leqslant r$.

The next corollary was first observed by Huneke (cf. [12]).

(1.9) COROLLARY. *If $H_0(I), ..., H_r(I)$ are Cohen–Macaulay for $r<\delta/2$, then I is (SCM).*

(1.10) DEFINITION. The *type* of I is equal to the minimum number of generators of the canonical module of $S=R/I$, i.e., $\operatorname{type}(I)=\nu(\operatorname{Ext}^g(S, R))$.

(1.11) EXAMPLE. Let $R=k[[x_1, ..., x_6]]$ be a power series ring in six variables over a field k, and consider the ideal I generated by the two by two minors of the generic symmetric matrix

$$\Phi=\begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix}.$$

I is a Cohen–Macaulay prime ideal of codimension three and deviation three (cf. [17]). Moreover I is syzygetic, satisfies (SD), and is not (SCM). In fact $\operatorname{depth}(H_1(I))=1$, $\operatorname{depth}(H_2(I))=3$, and $\operatorname{depth}(H_3(I))=3$ (cf. [16]).

(1.12) PROPOSITION. *Let R be a local Gorenstein ring, and let I be a CM ideal of height $g\geqslant 2$ generated by n elements. If I satisfies (SD), then $H_{n-g}(I)$ and $H_{n-g-1}(I)$ are CM.*

Proof. $H_{n-g}(I)$ is the canonical module of R/I, so it is CM. Since $\operatorname{depth}(Z_{n-g})\geqslant \min\{d, d-g+2\}$, the exact sequence

$$0\to Z_{n-g}\to K_{n-g}\to B_{n-g-1}\to 0$$

yields $\operatorname{depth}(B_{n-g-1})+1\geqslant \min\{d, d-g+2\}$, while the (SD) condition gives $\operatorname{depth}(Z_{n-g-1}) \geqslant d-g$. Therefore $\operatorname{depth}(Z_{n-g-1}) \geqslant d-g \geqslant \operatorname{depth}(H_{n-g-1}(I))$.

To complete the proof notice that from the exact sequence

$$0 \to B_{n-g-1} \to Z_{n-g-1} \to H_{n-g-1}(I) \to 0$$

we get $\operatorname{depth}(H_{n-g-1}(I)) = d - g$, which means that $H_{n-g-1}(I)$ is CM. ∎

2. Rees Algebras

The aim of this section is to describe the canonical module and the divisor class group of the Rees algebra $R(I)$ of a normal ideal I of the kind studied in Section 1.

2.1. *Divisor Class Group*

Let $I = (x_1, ..., x_n)$ be an ideal of the ring R. The mapping θ: $R^n \to I$ given by $\theta(z_1, ..., z_n) = z_1 x_1 + \cdots + z_n x_n$ induces an R-algebra epimorphism β: $R[T_1, ..., T_n] \to \operatorname{Sym}(I)$. The kernel of β is generated by all linear forms $\sum_{i=1}^n b_i T_i$ such that $\sum_{i=1}^n b_i x_i = 0$.

The *Rees algebra* $R(I)$ of I is the subring $R[x_1 T, ..., x_n T] \subset R[T]$. There exists an epimorphism ϕ: $R[T_1, ..., T_n] \to R(I)$, induced by $T_i \to x_i T$. The kernel of ϕ will be denoted by J, and is generated by all forms $F(T_1, ..., T_n)$ such that $F(x_1, ..., x_n) = 0$. In particular, we may factor ϕ through $\operatorname{Sym}_R(I)$ = symmetric algebra of I, and obtain the commutative diagram

$$\begin{tikzcd} R[T_1, ..., T_n] \arrow[r, "\phi"] \arrow[d, "\beta"] & R(I) \\ \operatorname{Sym(I)} \arrow[ur, "\alpha"'] & \end{tikzcd}$$

where α is an epimorphism. We say that I is an ideal of *linear type* if α is an isomorphism.

Let I be an ideal of a ring R, and denote by $B = R(I)$ its Rees algebra. The associated graded ring of I, $G = \operatorname{gr}_I(R) = \bigoplus_{i \geq 0} I^i/I^{i+1}$, can be written as B/IB. If R is a local ring then $\dim(G) = \dim(R)$ and $\dim(B) \leq \dim(R) + 1$, with equality if and only if I has positive height (cf. [3]).

(2.1.1) Lemma. *Let (R, m) be a Cohen–Macaulay local ring with infinite residue field k, and let I be an ideal of R of positive height. If G is Cohen–Macaulay then there exists an element x in $I - mI$ such that x^* the class of x in $G_1 = I/I^2$ is G-regular.*

Proof. Let $q_1, ..., q_n$ be the minimal primes of G, and denote the homogeneous component of degree one of q_i by q_{i1}. Consider the canonical map $\phi: G_1 \to G_1^*$, where $G_1^* = G_1/mG_1$; set $q_{i1}^* = \phi(q_{i1})$. We claim that q_{i1}^* are proper subspaces of G_1^*. Otherwise we have $G_1 = q_{i1} + mG_1$, from Nakayama's lemma, $G_1 = q_{i1}$. Hence G_+ is contained in q_i, which is impossible because $\mathrm{ht}(G_+) = \mathrm{ht}(I)$, while q_i is a minimal prime of G. To complete the proof notice that since k is infinite G_1 cannot be a finite union of proper subspaces. We may then pick an element $x \in G_1 - (q_{11} \cup \cdots \cup q_{n1} \cup mG_1)$. ▮

In the remark below we keep the assumptions and notations of the lemma.

(2.1.2) *Remarks.* (a) If x is a prime element of R, then (x, xt) is a prime ideal of $B = R[It]$.

(b) $(xt) = (x, xt) \cap B_+$ and $xB = (x, xt) \cap IB$.

Proof. (a) Since x^* is G-regular we have $(I^k : x) = I^{k-1}$ for $k \geq 1$. Therefore we get the exact sequence

$$0 \to xI^{k-1} \to I^k \xrightarrow{p} I^k + (x)/(x) \to 0.$$

Observe that p induces an exact sequence at the level of Rees algebras

$$0 \to (x, xt) \to R(I) \to R'(I') \to 0.$$

Here $R' = R/(x)$, and $I' = I/(x)$. Therefore (x, xt) is a prime ideal since R' is a domain.

(b) The second equality follows from $(I^k : (x)) = I^{k-1}$, while the first equality is easy to prove. ▮

Let M be an R-module, and let p be a prime ideal of the ring R. We set

$$T(p) = \ker(\mathrm{Sym}(M) \to \mathrm{Sym}(M) \otimes k(p)), \qquad \text{where } k(p) = Rp/pR_p.$$

(2.1.3) PROPOSITION. *Let I be an ideal of R, and let P be a minimal prime over IB. If I is of linear type then $P = T(p) = pB_p \cap B$, where $p = P \cap R$.*

Proof. We set $p = P \cap R$. From the isomorphism $R(I) \otimes k(p) \cong \mathrm{Sym}_{k(p)}(I_p/pI_p)$ we get $pB_p \in \mathrm{Spec}(B_p)$. Clearly $pB_p \subset PB_p$ and $I \subset p$, as $P \cap (R - p) = \varnothing$ we have $PB_p \cap B = P$. Therefore the minimality of P gives $P = T(p)$. ▮

Let S be a normal domain, and let J be a divisorial ideal of S. We denote by $Cl(S)$ the divisor class group of S, and by $[J]$ the ideal class of J. For basic results and notation we shall refer to [4, 15].

The main theorem of this section is the following.

(2.1.4) THEOREM. *Let (R, m) be a regular local ring with infinite residue field k. Assume that I is a Cohen–Macaulay ideal generated by a d-sequence and satisfies (SD). If $B = R(I)$ is integrally closed, then $Cl(B)$ is a free abelian group of finite rank.*

(2.1.5) *Remark.* Let (R, m) be a CM local domain, and let I be a CM ideal in R. Assume that I satisfies (SD). In [10] it is shown that if $v(I_p) \leqslant \mathrm{ht}(p)$ for each prime ideal p containing I, then both $R(I)$ and $\mathrm{gr}_I(R)$ are CM and I is generated by a d-sequence (see also [13]).

The following result is due to Nagata and will be used often.

(2.1.6) THEOREM. *Let T be a multiplicatively closed set of a normal domain R. Then there exists an exact sequence of groups*

$$0 \to H \to Cl(R) \to Cl(T^{-1}R) \to 0,$$

where H is the subgroup of $Cl(R)$ generated by the classes of height one prime ideals which contain elements of T. In particular if T is generated by prime elements then $Cl(R) = Cl(T^{-1}R)$.

Proof of Theorem (2.1.4). To begin choose an element x as in lemma (2.1.1). We may harmlessly assume that x is a prime element—by going over to a polynomial ring $R[Y]$ if necessary, a step that leaves unchanged the divisor class group. Using Nagata's theorem we obtain the exact sequence

$$0 \to H \to Cl(B) \to Cl(B_{xt}) \to 0,$$

where H is the subgroup of $Cl(B)$ generated by the classes of height one prime ideals which contain (xt). Since $(xt) = B_+ \cap (x, xt)$ and $IB \cong B_+$, we get $H = Z[IB]$.

We next determine $Cl(B_{xt})$. By Proposition (2.1.3) and the normality of $R(I)$, it follows that IB has a primary decomposition of the form

$$IB = T_1^{(e_1)} \cap \cdots \cap T_s^{(e_s)}, \qquad \text{where } T_i = T(p_i).$$

We apply Nagata's theorem once more to get the exact sequence

$$0 \to H_1 \to Cl(B_{xt}) \to Cl((B_{xt})_x) \to 0,$$

where H_1 is generated by the minimal primes over xB_{xt}.

Note that $IB_{xt} = xB_{xt}$; therefore localizing the primary decomposition of IB at xt we get $xB_{xt} = \bigcap_{i=1}^{i=s} (T_i)_{xt}^{(e_i)}$ which—together with the factoriality of $(B_{xt})_x = R_x[t, t^{-1}]$—implies that $Cl(B) = \langle [IB], [T_i] \rangle$, $i \in \{1, \ldots, s\}$. Since

$xB = IB \cap (x, xt)$, $Cl(B)$ is indeed generated by the set $\{[T_1], ..., [T_s]\}$. We now show that this is a linearly independent set. Assume

$$r_1[T_1] + \cdots + r_s[T_s] = 0,$$

We may order the set $\{p_1, ..., p_s\}$ so that p_i is a minimal element of the set $\{p_i, ..., p_s\}$. Let us show that $r_1 = 0$. By Nagata's theorem we have an exact sequence

$$0 \to H \to Cl(B) \to Cl(B_{p_1}) \to 0;$$

therefore the equation above, viewed in $Cl(B_{p_1})$, becomes $r_1[p_1 B_{p_1}] = 0$. It is easy to see that $[p_1 B_{p_1}]$ has infinite order, that is, $r_1 = 0$. By induction it follows that $r_1 = \cdots = r_s = 0$. ∎

Keeping the assumptions of the theorem above we obtain:

(2.1.7) COROLLARY. *Let I be a prime ideal of R different from m. Assume that I is a complete intersection on the punctured spectrum of R, and* $\dim(R) = v(I)$. *Then $Cl(B)$ is a free abelian group of rank two, and we can write it as*

$$Cl(B) = Z[IB] \oplus Z[mB].$$

Proof. Notice that $T(p)$ is a minimal prime over IB if and only if $v(I_p) = \mathrm{ht}(p)$; as a result $IB = T(I) \cap (mB)^{(e)}$. ∎

(2.1.8) EXAMPLE. (a) We consider the ideal of Example (1.11); in this case $R(I)$ is known to be normal (cf. [16]). Therefore the divisor class group of $R(I)$ can be expressed as

$$Cl(B) = Z[IB] \oplus Z[mB].$$

(b) Consider the ideal $= (x_1x_2, x_1x_5, x_1x_6, x_4x_5, x_3x_4, x_2x_3, x_6x_7)$ of the polynomial ring $R = Q[x_1, ..., x_7]$ (see [22]). Notice that $\mathrm{gr}_I(R)$ is CM, because $v(I_p) \leqslant \mathrm{ht}(p)$ for $p \in \mathrm{Spec}(R)$. Taking this into account the normality of $R(I)$ is not hard to check (see [2] for the normality criteria available): it suffices to show that for each minimal prime P of (I, J), the image of J in the C/P-module P/P^2 has rank equal to six($C = k[x_i, T_i]$, $i = 1, ..., 7$), and the rank condition is easy to show.

The primary decomposition of IB has the form $IB = T(p_1) \cap \cdots \cap T(p_8) \cap T(q)^{(e)} \cap (mB)^{(f)}$. Here $p_1 = (x_1, x_2, x_4, x_6)$, $p_2 = (x_1, x_2, x_4, x_7)$, $p_3 = (x_1, x_3, x_4, x_6)$, $p_4 = (x_1, x_3, x_4, x_7)$, $p_5 = (x_1, x_3, x_5, x_6)$, $p_6 = (x_1, x_3, x_5, x_7)$, $p_7 = (x_2, x_3, x_5, x_6)$, $p_8 = (x_2, x_4, x_5, x_6)$, and $q = (x_1, x_2, x_3, x_4, x_5, x_6)$. Therefore $Cl(B)$ is a free abelian group of rank ten.

2.2. Canonical Module

The purpose of this section is to describe the canonical module of the Rees algebra of certain deviation three ideals.

Throughout R will be a local ring of dimension d, I will be an ideal in R of height $g \geqslant 2$, minimally generated by n elements, and B will denote $R(I)$.

(2.2.1) PROPOSITION. *Let (R, m) be a local Gorenstein ring, and let I be a perfect prime ideal in R of deviation three and height g. If I is syzygetic and $n = d$, then the homology modules* $\mathrm{Ext}^{r-1}(Z_i, R)$ *can be determined as (set $r = n - 1$)*

$$\begin{array}{rl} 0 & \text{if } i = 0, 1, 3; \\ \mathrm{Ext}^d(I^{(2)}/I^2, R) & \text{if } i = 2; \\ R/I & \text{if } 4 \leqslant i < r; \\ R & \text{if } i = r. \end{array}$$

Proof. We first take care of the case $i = 2$. Consider the exact sequences

$$0 \to Z_2 \to K_2 \to B_1 \to 0$$

$$0 \to B_1 \to Z_1 \to H_1 \to 0.$$

Notice that $\mathrm{pd}_R(Z_1) = g - 2$; as a consequence we obtain

$$\mathrm{Ext}^{r-2}(Z_2, R) = \mathrm{Ext}^{r-1}(B_1, R) = \mathrm{Ext}^r(H_1, R).$$

On the other hand, since I is syzygetic we have the exact sequences

$$0 \to H_1(I) \to (R/I)^n \to I/I^2 \to 0$$

$$0 \to I^2 \to I \to I/I^2 \to 0.$$

As $\mathrm{pd}_R(R/I) = g$, we get

$$\mathrm{Ext}^r(H_1(I), R) \cong \mathrm{Ext}^{r+1}(I/I^2, R) \cong \mathrm{Ext}^r(I^2, R).$$

Using $\mathrm{pd}_R(I^{(2)}) \leqslant d - 2$ together with the exact sequence

$$0 \to I^2 \to I^{(2)} \to I^{(2)}/I^2 \to 0$$

gives $\mathrm{Ext}^r(I^2, R) = \mathrm{Ext}^{r+1}(I^{(2)}/I^2, R)$, which takes care of the case $i = 2$.

We now examine the case $n - g + 1 \leqslant i \leqslant n - 2$. The acyclic tail of the Koszul complex gives the following free resolution for Z_i:

$$K_*: 0 \to K_n \to \cdots \to K_{i+1} \to Z_i \to 0.$$

Therefore $\mathrm{Ext}^{r-i}(Z_i, R) \cong H^n(\mathrm{Hom}(K_*, R)) \cong R/I.$

The remaining cases, i.e., $i=0, 1, 3$, and r are easy to prove; it suffices to use $\operatorname{depth}(Z_{n-g})=d-g+2$. ∎

The main result of this section is the following:

(2.2.2) THEOREM. *Let (R, m) be a regular local ring of dimension d, and let I be a CM prime ideal of height g and deviation three. Assume I is generated by a d-sequence, is a complete intersection in codimension at most one, and $d=n$. If I satisfies (SD), then the canonical module of the Rees algebra of I can be written as (we set $B=R(I)$)*

$$W_{R(I)}=\left[\bigoplus_{i=0}^{g-2} Rt^i\right]\oplus It^{g-1}\oplus\left[\bigoplus_{i=0}^{\infty} I^{(2)}I^it^{g+i}\right].$$

The idea for its proof is to use a description of the module W_B/B_+W_B, and then lift it to W_B. We basically need the next two results.

(2.2.3) THEOREM (cf. [11]). *Let R be a CM local domain, and let I be an ideal in R minimally generated by n elements. Assume that for each prime ideal p containing I, $v(I_p)\leqslant \operatorname{ht}(p)$. If I satisfies (SD) then $\operatorname{Sym}(I)\cong R(I)$, and there is an isomorphism of graded B-modules (here we set $r=n-1$)*

$$W_B/B_+W_B\cong\bigoplus \operatorname{Ext}^{r-i}(Z_i, R).$$

(2.2.4) THEOREM (cf. [15]). *Let (R, m) be a regular local ring, and let I be an ideal in R of height g. Assume that I is generically a complete intersection. If $\operatorname{gr}_I(R)$ is a domain, or if I is (SCM), then the canonical module of $B=R(I)$ is given by*

$$W_B=(1, t)^{g-2}=\left[\bigoplus_{i=0}^{g-2} Rt^i\right]\oplus\left[\bigoplus_{i=1}^{\infty} I^it^{i+g-2}\right].$$

Proof of Theorem (2.2.2). As W_B is a graded $R(I)$-ideal, we may write $W_B=\bigoplus W_it^i$, with $W_i\subset R$. Using (2.2.4), we may also assume $W_{B_I}=(1, t)^{g-2}$.

Taking into account (2.2.1) and (2.2.3) we obtain that $W_k/\sum_{i=0}^{k-1} W_iI^{k-i}$ is isomorphic to

$$\begin{array}{rl} R/I & \text{if } 1\leqslant k\leqslant g-2;\\ \operatorname{Ext}^d(I^{(2)}/I^2, R) & \text{if } k=g;\\ 0 & \text{otherwise.}\end{array}$$

We may therefore write $W_0=aR$, with $a\notin I$. Assume $W_0=W_1=\cdots=W_k=aR$ for $0\leqslant k\leqslant g-3$; since $W_{k+1}/aI\cong R/I$, there is an element b in R so that $W_{k+1}=(aI, b)$, and $(aI:b)=I$. We claim that

$x = ba^{-1}$ is a unit in R. To see this pick a regular sequence $\{c, d\}$ in I; using that $xI \subset I$ and $c(dx) = d(cx)$ it follows that $x \in R$, so we can write $W_{k+1} = a(I, x)$. Pick now a prime ideal p of codimension $g+2$ containing (a, b, I). Because I is (SCM) in codimension $g+2$, (2.2.4) says that $(W_{k+1})_p = a(I, x)\, R_p \cong R_p$, and this implies that x is a unit in R. So, we have proved $W_k = aR$ for $0 \leqslant k \leqslant g-2$, and as a consequence $W_{g-1} = aI$.

Next we show the equality $W_g = aI^{(2)}$. Using that $\mathrm{Ext}^d(-, R)$ is a dualizing functor on the category of R-modules of finite length (cf. [7]), together with the isomorphism $W_g/aI^2 \cong \mathrm{Ext}^d(I^{(2)}/I^2, R)$, we obtain $W_g/aI^2 \cong I^{(2)}/I^2 \cong aI^{(2)}/aI^2$. Thus to prove $W_g = aI^{(2)}$ it suffices to show that $aI^2 \subset W_g \subset aI^{(2)}$. Observe that $m^r \cdot W_g \subset aI^2$ for some r, so that $aI^2 \subset W_g \subset (a) \cap I^{(2)}$. Since a is not in I it follows that $(a) \cap I^{(2)} = aI^{(2)}$.

To complete the proof, we use that $W_k = \sum_{i=0}^{k-1} W_i I^{k-i}$ for $k \geqslant g+1$ to get $W_{g+k} = aI^{(2)}I^k$. ∎

As a consequence of the proof above we have:

(2.2.5) Corollary. *$I^{(2)}/I^2$ is a Gorenstein module, that is, $\mathrm{Ext}^d(I^{(2)}/I^2, R) \cong I^{(2)}/I^2$.*

(2.2.6) *Remark.* Let I be the ideal of Example (1.11). Notice that the canonical module of I can be expressed as

$$W_B = R + Rt + It^2 + I^{(2)}t^3 + I^{(2)}It^4 + \cdots + I^{(2)}I^k t^{k+3} + \cdots.$$

There is a description for the symbolic powers of the ideal generated by the $(n-1)$-sized minors of a generic symmetric matrix ψ of order n (cf. [1]). For $n = 3$, this gives $I^{(2)} = (I^2, \Delta)$, where $\Delta =$ determinant of ψ. Therefore $R(I)$ has type 3.

3. Numerical Constraints on Koszul Homology

The purpose of this section is to show certain numerical constraints that exist on the projective rèsolution of graded ideals with Cohen–Macaulay Koszul homology. A drawback of some proofs is that they were carried by a computer package, in this case Macsyma. Hand calculations (and considerable patience) would—once the results are shown likely—serve the same purposes.

3.1. *Pure Resolutions*

Let $R = k[x_1, ..., x_n]$ be the polynomial ring with the usual grading and let M be a finitely generated, non-negatively graded R-module. By the resolution of M, we mean the minimal homogeneous resolution of M by free R-modules:

$$0 \to \bigoplus_{i=1}^{b_g} R(-d_{gi}) \to \cdots \to \bigoplus_{i=1}^{b_1} R(-d_{1i}) \to \bigoplus_{i=1}^{b_0} R(-d_{0i}) \to M \to 0.$$

The letters in ( ) are the twists; they indicate a shift in the graduation, e.g., $R(d)_i = R_{d+i}$.

(3.1.1) DEFINITION. M is said to have a *pure resolution* of type $(d_0, d_1, ..., d_g)$, where $d_0 < d_1 < \cdots < d_g$, if the minimal homogeneous resolution of M has the form

$$0 \to R^{b_g}(-d_g) \to \cdots \to R^{b_1}(-d_1) \to R^{b_0}(-d_0) \to M \to 0.$$

The integers $b_0, ..., b_g$ are the Betti numbers of M. If $d_i = d_1 + i$, $2 \leqslant i \leqslant g$, the resolution is said to be *linear*. We remark that we may—and shall henceforth do—assume that $d_0 = 0$. Only cyclic modules will play the role of M.

(3.1.2) According to Herzog and Kühl (cf. [8]), Cohen–Macaulay modules with pure resolutions are precisely those modules with Betti numbers satisfying

$$b_i = \left(\prod_{j \neq i} \frac{d_j}{|d_j - d_i|} \right) b_0.$$

Since the Betti numbers and the twists are positive integers, these formulae impose severe restrictions on the numbers $d_1 < d_2 < \cdots < d_g$ that can occur as the resolution type of a CM module with a pure resolution.

(3.1.3) Using the formulae above, Huneke and Miller (cf. [9]) have expressed the multiplicity of a CM module with a pure resolution entirely in terms of the twists:

$$e(M) = \left(\prod_{i=1}^{g} d_i \right) b_0 / g!.$$

Let S be a graded quotient R/I of the polynomial ring $R = k[x_1, ..., x_n]$ over an infinite field k. (In fact this last hypothesis is not essential since a change of the coefficient field could easily be carried out while preserving the requirement on I, and leaving dimensions of the vector spaces of forms of a given degree in S or in an S-module unchanged.) We will study those S with pure resolution with the aim of attempting to understand the scarcity of classes of examples of ideals with Cohen–Macaulay Koszul homology. That is, despite the fact that such ideals exist in profusion, they seem to be located in well circumscribed classes. We shall be focusing in codimension 3, although the methods have a wider validity as indicated in Section 3.2.

Several numerical computations have put in evidence and support the following conjecture.

(3.1.4) *Conjecture.* Let I be a homogeneous ideal of R of codimension 3, and deviation at least 3. Assume that I is not a Gorenstein ideal and is generically a complete intersection. If the resolution of S is pure then I is not (SCM).

The part of the conjecture proved—likely more than half of all possible cases—is:

(3.1.5) THEOREM. *Let I be a codimension 3 Cohen–Macaulay ideal as above with a resolution*

$$0 \to R^{b_3}(-(d+a+b)) \to R^{b_2}(-(d+a)) \to R^c(-d) \to I \to 0.$$

If $a \geqslant b$ and $c \geqslant 6$, then $H_1(I)$, the first Koszul homology module, is not Cohen–Macaulay.

Our approach to its proof will be computational. That is, we shall compare the expected multiplicity of $H_1(I)$ to partial Hilbert functions sums derived from the resolution of I. Our starting point is a criterion of Herzog [6].

(3.1.6) PROPOSITION. *Let S be a Cohen–Macaulay local ring and let M be a finitely generated S-module with a well-defined and positive rank, and let y be a system of parameters for S. Then*

$$l(S/(y)) \cdot \operatorname{rank}(M) \leqslant l(M/(y)M).$$

Furthermore equality holds if and only if M is Cohen–Macaulay.

Proof of (3.1.5). Because I is generically a complete intersection the module $H_1 = H_1(I)$ has a well-defined rank, equal to the deviation $(\mathrm{I}) = c-3$. Thus, according to (3.1.6), it suffices to prove that for some system of parameters y, $l(H_1(I)/(y)\, H_1(I)) > \operatorname{rank}(H_1(I) \cdot l(S/(y)S))$.

We start by making a specialization to the case of a polynomial ring of dimension three. Since k is infinite, there exists a system of parameters $y = \{y_1, ..., y_{n-3}\}$ for S with each y_i a form of degree one of R. We make two observations: (i) Because $\operatorname{Tor}_i(S, R/(y)) = 0$ for $i \geqslant 1$, it is clear that the minimal resolution of $S^* = S/(yS)$ as an R^* $(=R/(y))$ -module has the same twists and Betti numbers as the minimal resolution of S over R. (ii) With the H_i's Cohen–Macaulay it follows from [20] that the Koszul homology modules of $I \otimes R^*$ over R^* are precisely $H_i(I) \otimes R^*$. We may then in the sequel assume that S is zero-dimensional.

We now begin the task of comparing the integer $e(H_1) = \text{rank}(\text{original } H_1(I)) \cdot e(S) = (c-3) \cdot e(S)$, with partial Hilbert sums contributing to $l(H_1)$. We first note that the length of S can be calculated using (3.1.3):

$$e(S) = l(S) = \frac{d(d+a)(d+a+b)}{3!}.$$

Consider the exact sequences

$$0 \to R^{b_3}(-(d+a+b)) \to R^{b_2}(-(d+a)) \to Z_1 \to 0$$
$$0 \to Z_2 \to R^{\binom{c}{2}}(-2d) \to B_1 \to 0.$$

If we denote by $l(M)_i$ the dimension of the ith component of the graded module M we may write

$$l(H_1)_i = l(Z_1)_i - l(B_1)_i = l(Z_1)_i - \binom{c}{2} l(R(-2d))_i + l(Z_2)_i.$$

More explicitly

$$l(H_1)_i = b_2 \binom{i-(d+a)+2}{2} - b_3 \binom{i-(d+a+b)+2}{2}$$
$$-\binom{c}{2}\binom{i-2d+2}{2} + l(Z_2)_i.$$

(3.1.7) The first Betti number is given by the formula

$$c = \frac{(d+a)(d+a+b)}{a(a+b)}.$$

Solving for d^2 we get the equation

$$d^2 = -(2a+b)d + (c-1)a^2 + (c-1)\,ab.$$

Using this equation we can write the remaining Betti numbers of S as

$$b_2 = (-d + (c-1)a + (c-1)b)/b$$
$$b_3 = (-d + (c-1)a)/b$$

as well as the multiplicity of S

$$e(S) = a(a+b)\,cd/6.$$

We define the cubic polynomial

$$Q(x) = b_2\binom{x-(d+a)+3}{3} - b_3\binom{x-(d+a+b)+3}{3}$$
$$-\binom{c}{2}\binom{x-2d+3}{3} - l(S)\,\mathrm{rank}(H_1(I)),$$

which expanded has the expression

$$\begin{aligned}Q(x) = {} & -((c^2-3c+2)x^3 + ((12c-6c^2)d + 6c^2 - 18c + 12)x^2\\
& + (((-12b-24a-24)c^2 + (18b+36a+48)c)d + (12ab+12a^2)c^3\\
& + (-30ab - 30a^2 + 11)c^2 + (18ab + 18a^2 - 33)c + 22)x\\
& + ((-8ab - 8a^2)c^3\\
& + (-8b^2 + (-12a-24)b - 12a^2 - 48a - 22)c^2\\
& + (10b^2 + (22a+36)b + 22a^2\\
& + 72a + 44)c)d + (8ab^2 + (24a^2 + 24a)b + 16a^3 + 24a^2)c^3\\
& + (-18ab^2 + (-54a^2 - 60a)b - 36a^3 - 60a^2 + 6)c^2\\
& + (10ab^2 + (30a^2 + 36a)b + 20a^3 + 36a^2 - 18)c + 12)/12.\end{aligned}$$

If x is a positive integer an application of the identity

$$\sum_{i=0}^{i=N} l(R)_i = \binom{N+3}{3}$$

leads to the equality

$$\sum_{i=0}^{i=x} l(H_1)_i - l(Z_2)_i = Q(X) + e(S)\,\mathrm{rank}(H_1(I)).$$

Using this equality the proof reduces to showing that $Q(z) > 0$ for some positive integer z greater or equal than $2d$. Our choice is that of the largest positive integer i for which $l(H_1)_i - l(Z_2)_i > 0$. One verifies directly that this number is the integral part of

$$z_0 = \frac{\sqrt{8c(c-2)(c-3)d^2 + (c-1)^2(c-2)^2} + 4c(c-2)\,d - 3(c-1)(c-2)}{2(c-1)(c-2)}.$$

Note that $Q(x)$ is a polynomial of degree 3 whose leading coefficient is negative. We may assume that $Q(2d) \leqslant 0$, as $Q(2d) > 0$ already implies that $H_1(I)$ is not Cohen–Macaulay.

(3.1.8) We set $P = (6(c-2)(c-1)^2 Q(z_0))/c$. The expression for P is

$$\begin{aligned}P = {} & (c-3)d^2 sqrt(8(c-3)(c-2)cd^2 + (c-2)^2(c-1)^2)\\ & + (c-2)((b^2+5ab+5a^2)c^2 - (4b^2+26ab+26a^2)c\\ & - 5b^2 - 11ab - 11a^2)d - a(a+b)(b+2a)(c-5)(c-2)(c-1)(c+1).\end{aligned}$$

Another observed fact is that $Q(z_0) = Q(z_0 - 1)$ (this uses the graph of the cubic). To sum up, we must show that if $a \geqslant b$ and $c \geqslant 6$, then P is positive.

We write $P = E\sqrt{W} - F$; using (3.1.7), it follows that E and F are positive for $a, b, \geqslant 1$, and $c \geqslant 6$. Consequently P and $E^2W - F^2$ have the same sign. Because of (3.1.7), $E^2W - F^2$ is a polynomial in d of degree one. It admits a representation

$$E^2W - F^2 = (c-2)(c-1)^2(-(2a+b)\,dH + a(a+b)(c-1)L),$$

where H and L are the polynomials

$$\begin{aligned}H = {} & (-c^3+16c^2-61c+50)b^4 + (-2c^4+40c^3-126c^2-48c+120)ab^3\\ & + ((12c^4+29c^3-332c^2+5c+142)a^2 + c^3 - 8c^2 + 21c - 18)b^2\\ & + ((28c^4-22c^3-412c^2+106c+44)a^3\\ & + (2c^4-14c^3+26c^2+6c-36)a)b\\ & + (14c^4-11c^3-206c^2+53c+22)a^4 + (2c^4-14c^3+26c^2+6c-36)a^2\\ L = {} & (-c^3+16c^2-61c+50)b^4 + (-c^4+23c^3-49c^2-159c+170)ab^3\\ & + ((3c^4+58c^3-265c^2-152c+212)a^2 + c^3 - 8c^2 + 21c - 18)b^2\\ & + ((8c^4+70c^3-432c^2+14c+84)a^3 + (c^4-5c^3-3c^2+45c-54)a)b\\ & + (4c^4+35c^3-216c^2+7c+42)a^4 + (c^4-5c^3-3c^2+45c-54)a^2.\end{aligned}$$

At this point we explain how the positivity of several polynomials was decided. It simply involved the examination of the coefficients of the polynomials over the variables a and b. To various ranges of c a matching of positive and negative coefficients was carried out. The key hypothesis always turned out to be "$a \geqslant b$," since it would ensure that an expression $pa^n + qa^r b^{n-r}$ would be positive as long as $p+q$ is positive.

Using this method, it was easy to see that H and L are positive for $a \geqslant b$, and $c \geqslant 6$. Solving for d in the equation in (3.1.7) yields

$$d = \frac{-(2a+b) + \sqrt{4a^2c + 4abc + b^2}}{2}.$$

Therefore $E^2W - F^2$ will be positive if and only if the following polynomial is positive in the indicated range:

$$N = (2a(a+b)(c-1)L + (2a+b)^2 H)^2 - (2a+b)^2 (4a^2c + 4abc + b^2) H^2.$$

As written out this polynomial covers the length of three full pages. Fortunately it turns out that it admits a factorization

$$N = 4(c-1)^4 a^4 (a+b)^4 G,$$

where G is the polynomial

$$\begin{aligned} G = {} & (c-4)^2 (c-2)(c^3 - 16c^2 + 61c - 50) b^4 \\ & - 2(c-4)(c-2)(c+1)(4c^3 - 27c^2 + 41c + 10)\, ab^3 \\ & + (8c^6 - 154c^5 + 1073c^4 - 3388c^3 + 4717c^2 - 2052c - 156)\, a^2b^2 \\ & - (c-3)^2 (c-2)(2c^3 - 26c^2 + 93c - 82) b^2 \\ & + 2(16c^6 - 248c^5 + 1441c^4 - 3822c^3 + 4349c^2 - 1356c + 4)\, a^3b \\ & + 2(c-3)^2 (c-2)(c+1)(4c^2 - 17c + 2)\, ab \\ & + (16c^6 - 248c^5 + 1441c^4 - 3822c^3 + 4349c^2 - 1356c + 4) a^4 \\ & + 2(c-3)^2 (c-2)(c+1)(4c^2 - 17c + 2) a^2 + (c-3)^4 (c-2)^2. \end{aligned}$$

One verifies that all the terms of G—viewed as a polynomial in the variables a and b—are positive except for two of them. Pairing them with positive terms it is easy to see that if $c \geqslant 11$ then G is positive. The remaining cases are then easily screened out by evaluation. ∎

3.2. *Concluding Remarks*

(3.2.1) *Remark.* We have ignored the contribution of the Hilbert function of Z_2. This provides further evidence in support of the conjecture. Let us indicate how to get a sharper bound for the length of $H_1(I)$.

Given the minimal presentation of Z_1, there exist exact sequences [23]

$$\begin{aligned} 0 & \to R^A(-2(d+a+b)) \to R^B(-(2d+2a+b)) \\ & \to R^C(-2(d+a)) \to \Lambda^2 Z_1 \to 0, \end{aligned}$$

where

$$A = \binom{b_3+1}{2}, \qquad B = b_2 b_3, \qquad C = \binom{b_2}{2},$$

and

$$0 \to \Lambda^2 Z_1 \to Z_2 \to D \to 0.$$

Therefore we obtain

$$\begin{aligned} l(H_1)_x &= l(Z_1)_x - l(K_2)_x + l(\Lambda^2 Z_1)_x + l(D)_x \\ &= b_2 \binom{x-(d+a)+2}{2} - b_3 \binom{x-(d+a+b)+2}{2} - \binom{c}{2}\binom{x-2d+2}{2} \\ &\quad + \binom{b_3+1}{2}\binom{x-2(d+a+b)+2}{2} - b_2 b_3 \binom{x-(2d+2a+b)+2}{2} \\ &\quad + \binom{b_2}{2}\binom{x-2(d+a)+2}{2} + l(D)_x. \end{aligned}$$

To illustrate its use, suppose there exists a ring S with a pure resolution

$$0 \to R^3(-15) \to R^{27}(-10) \to R^{25}(-9) \to R \to S \to 0.$$

Using the formula above to count lengths up to an optimal degree we obtain $l(H_1) \geqslant \sum_{i=0}^{21} (l(H_1)_i - l(D)_i) = 4980 > 4950 = \operatorname{rank}(H_1(I)) \cdot e(S)$. Therefore (by Proposition (3.1.6)) $H_1(I)$ could not be Cohen–Macaulay.

(3.2.2) *Remark.* Kustin, Miller, and Ulrich [19] proved that for the ideals we are considering the condition $d < a+b$ forces I to be a Gorenstein ideal, and consequently is (SCM). Since Cohen–Macaulay ideals of deviation at most 2 are always (SCM), we may then look at examples where the conditions $d \geqslant a+b$ and $c \geqslant 6$ are both present.

Consider the case $d = a+b$; it is easy to see that $a = b$ and that the resolution of S has the form

$$0 \to R^3(-4a) \to R^8(-3a) \to R^6(-2a) \to R \to S \to 0.$$

We can do better by only assuming $a = b$; an application of (3.1.2) yields the resolution

$$0 \to R^{\binom{n}{2}}(-(n+1)a) \to R^{n^2-1}(-na) \to R^{\binom{n+1}{2}}(-(n-1)a) \to R \to S \to 0.$$

Using Theorem (3.1.5) we see that $H_1(I)$ cannot be Cohen–Macaulay for $n \geqslant 3$.

There are examples of ideals with this kind of resolution. For instance, consider the ideal generated by the submaximal minors of a generic symmetric matrix of order n, $n \geqslant 3$. It can be shown that depth $H_1(I) = \binom{n+1}{2} - 5$; see [24].

(3.2.3) *Remark.* The formula (3.1.8) can also take care of certain cases with $b > a$, in fact, taking into account the integrality formulae of [8], one can show the validity of the conjecture for all cases in the range $6 \leqslant c \leqslant 24$, with the possible exception of $c = 8$. The ideal I of [19, Example 2.6] has the numerical data

$$0 \to R^2(-6) \to R^9(-4) \to R^8(-3) \to R \to R/I \to 0.$$

$H_1(I)$ is Cohen–Macaulay, as proved in [19, Example 2.6]. In fact, it is proved in [23] that CM ideals of type 2 (and height 3) always have $H_1(I)$ Cohen–Macaulay. Here we show that CM ideals with these twists, which are generically complete intersection, cannot have $H_2(I)$ CM. As before, if $H_2(I)$ is Cohen–Macaulay, after a specialization to a polynomial ring in three variables, the expected length of $H_2(I)$ is equal to rank$(H_2(I))$ $e(S) = 120$; we confront this with a lower estimate of $l(H_2)$. Consider the equality

$$\begin{aligned} l(H_1)_x &= l(Z_1)_x - l(B_1)_x = l(Z_1)_x - l(K_2)_x + l(Z_2)_x \\ &= 9\binom{x-2}{2} - 30\binom{x-4}{2} + l(Z_2)_x. \end{aligned}$$

It yields

$$\begin{aligned} l(H_2)_x &= l(Z_2)_x - l(B_2)_x = l(Z_2)_x - l(K_3)_x + l(Z_3)_x \\ &\geqslant 30\binom{x-4}{2} - 9\binom{x-2}{2} - 56\binom{x-7}{2}. \end{aligned}$$

To complete the argument notice that $\sum_{x=7}^{x=10} l(H_2)_x \geqslant 130 > l(H_2)$.

(3.2.4) *Remark.* From [8] it is possible to derive formulae for the twists that may occur in (3.1.5). Although this method works quite well for more general cases, we consider only the case $b = 1$, $c = 6$. (The question is which a's can occur.) If we set $x = a + 1$, we must solve the diophantine equation

$$6x(x-1) = y(y-1).$$

Its solution, as shown to us by D. Rohrlich, are given by

$$x = (X+1)/2 \qquad \text{and} \qquad y = (Y+1)/2,$$

where

$$\sqrt{6}X + Y = \pm(1 \pm \sqrt{6})(5 + 2\sqrt{6})^n, \qquad n \in Z.$$

(3.2.5) *Remark.* Let I be a graded ideal of R. Assume that $S = R/I$ is an algebra of codimension g ($g \geqslant 3$) having a linear resolution

$$0 \to R^{b_g}(-(d+g-1)) \to \cdots \to R^{b_2}(-(d+1)) \to R^c(-d) \to R \to S \to 0.$$

We expect that CM algebras of this type are not (SCM). For $g = 3$, this follows from the earlier result. The codimension four case is also easy to settle. That is, if I is generically a complete intersection then H_1 cannot be Cohen–Macaulay. We may, of course, assume that $d > 1$.

As before, we will use a specialization and assume that R is a polynomial ring in four variables. Proceeding as in the proof of (3.1.5) we obtain the equality

$$\begin{aligned} F &= \sum_{i=0}^{2d} l(H_1)_i - l(Z_2)_i \\ &= b_2\binom{d+3}{4} - b_3\binom{d+2}{4} + b_4\binom{d+1}{4} - \binom{c}{2}. \end{aligned}$$

Since R is a four-dimensional polynomial ring, the equality above is computed using the formula

$$\sum_{i=0}^{N} l(R_i) = \binom{N+4}{4}.$$

Notice also that in this case the rank of $H_1(I)$ is $c-4$. If, however, we evaluate $F - \operatorname{rank}(H_1(I)) \cdot e(S)$ we obtain

$$\begin{aligned} l(H_1) - \operatorname{rank}(H_1(I)) \cdot e(S) &\geqslant F - \operatorname{rank}(H_1(I)) \cdot e(S) \\ &= (d^2 + 10d + 6)(d-1)\, d(d+1)(d+2)/72 > 0. \end{aligned}$$

So by Proposition (3.1.6), $H_1(I)$ cannot be CM.

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